Lecture No.-24

Functions With Periods Other Than 2π

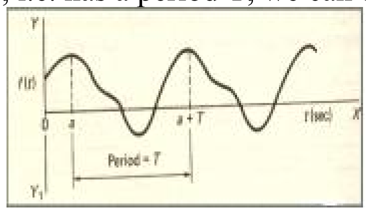
So far, we have considered functions f(x) with period 2π. In practice, we often encounter 2 2 functions defined over periodic intervals other than 2π, e.g. from 0 to T, $-\frac{T}{}$ to $\frac{T}{}$

etc.

Functions With Period T

2 If y = f(x) is defined in the range $-\frac{T}{}$

to $\frac{T}{}$, i.e. has a period T, we can convert this to an



2 interval of 2π by changing the units of the independent variable. In many practical cases involving physical oscillations,
the independent variable is time (t) and the periodic interval is normally denoted by T, i.e.

$$f(t) = f(t + T)$$

T Each cycle is therefore completed in T seconds and the frequency f hertz (oscillations per second) of the

periodic function is therefore given by $f = \frac{1}{}$. If the

angular velocity, ω radians per seconds, is defined by ω = 2πf, then

$$T\omega \quad \omega = \frac{2\pi}{}$$

and $T = \frac{2\pi}{}$

The angle, x radians, at any time t is therefore x = ωt and the Fourier series to represent the function can be expressed **as**

$$\mathbf{f(t) = \frac{1}{2} a_0 + \sum_{x=1}^{\infty} \{a_n Cos\ n\omega t + b_n Sin\ n\omega t\}}$$

which can also be written in the form

$$\mathbf{f(t) = \frac{1}{2} A_0 + \sum_{x=1}^{\infty} B_n \ sin\ (n\omega t + \phi_n)} \qquad n = 1, 2, 3, ...$$

Fourier Coefficients

With the new variable, the Fourier coefficients become:

$$f(t) = \frac{1}{2} a_0 + \sum_{x=1}^{\infty} \{a_n\ cosn\omega t + b_n\ sinn\omega t\}$$

T

2π/ω

$$a_0 = \frac{2}{T}\int_0^T f(t)\,dt = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t)\,dt$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega t\,dt = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t)\cos n\omega t\,dt$$

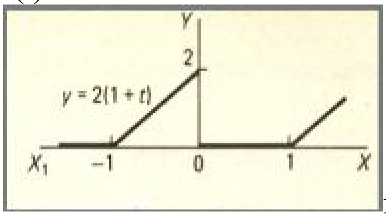
$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega t\,dt = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t)\sin n\omega t\,dt$$

We can see that there is very little difference between these expressions and those that have gone before. The limits can, of course, be 0 to T, $-\frac{T}{2}$ to $\frac{T}{2}$, $-\frac{\pi}{\omega}$ to $\frac{\pi}{\omega}$, 0 to $\frac{2\pi}{\omega}$ etc. as is convenient, so long as they cover a complete period.

Example

Determine the Fourier series for a periodic function defined by

$f(t) = 2(1+t) \quad -1 < t < 0$
$f(t) = 0 \quad 0 < t < 1$



$f(t) = f(t+2)$

The first step is to sketch the wave which is.

$$f(t) = \frac{1}{2}a_0 + \sum_{x=1}^{\infty}\{a_n \cos n\omega t + b_n \sin n\omega t\})$$

$T = 2$

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,dt = \frac{2}{2}\int_{-1}^{1} f(t)\,dt = \int_{-1}^{0} 2(1+t)\,dt + \int_{0}^{1} (0)\,dt = \left[2t + t^2\right]_{-1}^{0} = 1$$

$\therefore \quad a_0 = 1$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\omega t\,dt = \frac{2}{2}\int_{-1}^{1} f(t)\cos n\omega t\,dt = \int_{-1}^{0} 2(1+t)\cos n\omega t\,dt$$

$$a_n = 2\left\{\left[(1+t)\frac{\sin n\omega t}{n\omega}\right]_{-1}^{0} - \int_{-1}^{0}\frac{\sin n\omega t}{n\omega}\,dt\right\}$$

$$= 2\left\{(0-0) - \left[-\frac{\cos n\omega t}{n^2\omega^2}\right]_{-1}^{0}\right\}$$

$$= \frac{2}{n^2\omega^2}(1 - \cos n\omega)$$

Now $T = \frac{2\pi}{\omega}$

$\therefore \omega = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi \quad \therefore a_n = \frac{2}{n^2\omega^2}(1 - \cos n\pi)$

$\therefore \quad a_n = 0$ (n even)

$= \frac{4}{n^2\omega^2}$

Now for b_n

(n odd)

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin n\omega t\,dt$$

$$b_n = 2\int_{-1}^{0} 2(1+t)\sin n\omega t\,dt = 2\left\{\left[-(1+t)\frac{\cos n\omega t}{n\omega}\right]_{-1}^{0} + \frac{1}{n\omega}\int_{-1}^{0}\cos n\omega t\,dt\right\}$$

$$= 2\left\{-\frac{1}{n\omega} + \frac{1}{n\omega}\left[\frac{\sin n\omega t}{n\omega}\right]_{-1}^{0}\right\} = 2\left\{-\frac{1}{n\omega} + \frac{1}{n^2\omega^2}(\sin n\omega)\right\}$$

As before $\omega = \pi$

$\therefore \quad b_n = -\frac{2}{n\omega}$

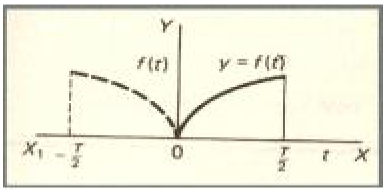
So the first few terms of the series give

$$f(t) = \frac{1}{2} + \frac{4}{\omega^2}\left\{\cos\omega t + \frac{1}{9}\cos 3\omega t + \frac{1}{25}\cos 5\omega t + ..\right\} - \frac{2}{\omega}\left\{\sin\omega t + \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t ..\right\}$$

Half-Range Series

The theory behind the half-range sine and cosine series still applies with the new variable.

(a) Even function



2 Half-range cosine series y

= f(t) 0 < t < T

f(t) = f(t + T)

symmetrical about the y-axis.

With an even function, we know that $b_n = 0$

$\therefore$ f(t) = $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}$

$a_n \cos n\omega t$

T/2

T where $a_0 = \frac{4}{}$

f(t) dt and $a_n = \frac{4}{} \int$

0 0

f(t) cos nωt dt

(b) Odd function Half-range sine series

2 y = f(t) 0 < t < T

f(t) = f(t + T)

symmetrical about the origin.

$\therefore a_0 = 0$ and $a_n = 0$

$$f(t) = \sum_{x=1}^{\infty} b_n \sin n\omega t;$$

T/2

T $b_n = \frac{4}{} \int$

0

f(t) sin nωt dt

Example

A function f(t) is defined by

f(t) = 4 − t, 0 < t < 4.

We have to form a half-range cosine series to represent the function in this interval.

First we form an even function, i.e. symmetrical about the y-axis.

T/2

$a_0 = \frac{4}{} \int$

$f_1(t)\,dt = \frac{4}{}$

$\int_4^4 (4 - t)\,dt = \frac{1}{}$

4

$\int (4 - t)\, dt$

T 0 8 0 4

$\frac{1}{}$ | t^2 | $\frac{1}{}$ [

2 0

$\frac{(4)^2}{}$] $\frac{1}{}$ $\frac{1}{}$

$= \frac{}{2} \left| 4t - \frac{}{2} \right| = \frac{}{2}$

T/2

0

| [4(4) –

2 |] $= \frac{}{2} [16 - 8] = \frac{}{2} (8) = 4$

4

$T a_n = \frac{4}{} \int$

0

$f_1(t) \cos n\omega t\, dt = \frac{4}{}$

$\int (4 - t) \cos n\omega t\, dt$

0

Simple integration by parts gives

$$a_n = \frac{1}{2} \left\{ -\frac{2 \sin 4n\omega}{n\omega} - \frac{1}{n^2\omega^2} (\cos 4n\omega - 1) \right\}$$

T = = 8 4 But $\omega = \frac{2\pi}{} \quad \frac{2\pi}{} \quad \frac{\pi}{}$

$$\therefore\ a_n = \frac{1}{2} \left\{ -\frac{2 \sin n\pi}{n\omega} - \frac{1}{n^2\omega^2} (\cos n\pi - 1) \right\}$$

n = 1, 2, 3, ……..

sin nπ = 0;

cos nπ = 1 (n even); cos nπ = – 1 (n odd)

∴ $a_n = 0$ (n even)

$a_n = \frac{1}{n^2\omega^2}$

⊥ ⌈

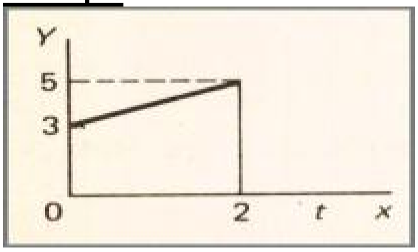
(n odd)

1 1 ⌉ π

∴ f(t)=2+ ω^2 { cos ωt+ 9cos3ωt+ 25cos5ωt+…} where ω = 4 .

⌊ ⌋

Example



A function f(t) is defined by

f(t) = 3 + t 0 < t < 2

f(t) = f(t + 4)

Obtain the half-range sine series for the function in this range.

Sine series required. Therefore, we form an odd function, symmetrical about the origin.

$a_0 = 0$; $a_n = 0$; T = 4

∞

f(t) = ∑

n=1

b_n sin nωt

T/2 2

T ∴ b_n = 4 ∫

0

f(t) sin nωt dt= ∫

0

(3 + t) sin nωt dt

–| (3+t)cosnωt| 2

|∫ (3 + t) sin nωt dt = |

0

– n ω

| – ∫

| 0 0

cos nωt

dt

– nω

(3 + 2) cos nω2 3 1 | sin nωt| 2 3 5

= 1 ⌈ sin 2nω+ | ⌉

– nω – nω nω

n ω | = nω – nω cos 2 nω+ $n^2\omega^2$ | nω – 0|

ω But T = 2π

| | 0 ⌊ ⌋

T=2 ∴ ω = 2π π

= 3

$-\frac{5}{}$

$\cos n\pi + \frac{1}{n\omega}\left[\frac{\sin n\pi}{n\omega}\right]$

$n^2\omega^2$

$n\omega \rfloor$

$\lfloor \mid b_n = \frac{1}{}(3 - 5\cos 2n\omega) + \frac{1}{}(\sin 2n\omega)$

$n\omega$

$\therefore\ b = \frac{1}{}$

$n^2\omega^2$

$)+ (3 - 5\cos n\pi \qquad \frac{1}{}(\sin n\pi) = -\frac{2}{}$

(n even)

$n \quad n\omega$

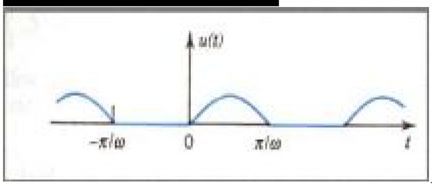
$n^2\omega^2$

$n\omega$

$= \omega \frac{8}{} \qquad$ (n odd) n

$\therefore f(t) = \frac{2}{\omega}\left\{4\sin\omega t - \frac{1}{2}\sin 2\omega t + \frac{4}{3}\sin 3\omega t - \frac{1}{4}\sin 4\omega t..\right\}$

Half-Wave Rectifier



A sinusoidal voltage E sin ωt, where t is time, is passed through a half-wave rectifier that clips the negative portion of the wave

Find the Fourier series of the resulting periodic functions.

$u(t) = 0$ if $\quad -T/2 < t < 0$

$\omega = E\sin\omega t \quad 0 < t < T/2$ here

$T = \frac{2\pi}{}$

$T a_0 = \frac{2}{}$

$\int_{-T/2}^{T/2}$

$u(t)\, dt = \frac{2}{}$

π/ω

0

$\int$

$-T/2$

$0\, dt + \frac{2}{}$

$\int_{0}^{T/2}$

$E \sin \omega t\, dt = \frac{2}{}$

$\int_{0}^{T/2}$

$E \sin \omega t\, dt = \frac{\omega}{} \int$

π/ω

0

$E \sin \omega t\, dt$

$= \frac{\omega}{\pi} E \left| \frac{-\cos \omega t}{\omega} \right| = \frac{2E}{\pi}$

0

$T/2$

$T/2$

π/ω

$T a_n = \frac{2}{}$

$\int_{-T/2}$

π/ω

$u(t) \cos n \omega t\, dt = \frac{2}{} \int$

0

$E \sin \omega t \cos n \omega t\, dt = \frac{\omega E}{}$

$\pi 2 \quad 0$

$2 \sin \omega t \cos n \omega t\, dt$

$= \frac{\omega E}{2\pi} \quad 0$

$[\sin (1+n) \omega t + \sin (1-n) \omega t]\, dt$

ωω If n = 1 then integral on the right is zero and if n = 2, 3, …… then we obtain.

$a_n = \frac{\omega E}{2\pi}\left[-\frac{\cos(1+n)\omega t}{(1+n)\omega} - \frac{\cos(1-n)\omega t}{(1-n)\omega} \right]_0^{\pi/\omega} = \frac{E}{2\pi}\left[\frac{-\cos(1+n)\pi + 1}{(1+n)\omega} + \frac{-\cos(1-n)\pi + 1}{(1-n)\omega} \right]$

$= \frac{\omega E}{2\pi\omega}\left[\frac{-\cos(1+n)\pi + 1}{(1+n)} + \frac{-\cos(1-n)\pi + 1}{(1-n)} \right]$

if n is odd then $a_n = 0$

if n is even then $a_n = \frac{E}{2\pi}\left(\frac{2}{1+n} + \frac{2}{1-n} \right) = \frac{E}{2\pi}\left[\frac{2-2n+2+2n}{(1+n)(1-n)} \right] = \frac{2E}{(1-n^2)\pi}$

$= \frac{2E}{(1-n)(1+n)\pi}$

For b_n we have

$a_n = \frac{2}{T}\int_{-T/2}^{T/2} u(t)\sin n\omega t\, dt = \frac{2}{T}\int_0^{\pi/\omega} E \sin\omega t \sin n\omega t\, dt = -\frac{\omega E}{2\pi}$

$-\,2\sin\omega t\sin n\omega t\,dt$

π $= -\dfrac{\omega E}{2}$

If n = 1

$\int_0^{\pi/\omega}$

$[\cos(1+n)\omega t-\cos(1-n)\omega t]\,dt$

$b_n = -\dfrac{\omega E}{}$

$\int^{\pi/\omega}$

$[\cos 2\omega t-1]\,dt = -\dfrac{\omega E}{}$

$\left|\dfrac{\sin 2\omega t}{}\right| - t\ {}^{\pi/\omega}$

$= -\dfrac{\omega E}{}$

$(-\pi/\omega) = E/2$

2π $\quad {}_0 \quad$ 2π

if $\quad n \neq 1$

$2\omega \qquad 2\pi$

$\dfrac{\omega E}{} \dfrac{\omega E}{}\left[\dfrac{\sin(1+n)\,\omega t}{}\right.$ $\qquad \pi/\omega \qquad \pi/\omega$

$-\quad -\sin(1\ n)\ t\Big]$

$0\Big|\ b_n = -\dfrac{}{2\pi}\int$

$[\cos(1+n)\omega t-\cos(1-n)\omega t]\,dt = -\dfrac{}{2\pi}\Big|$

$(1+n)\omega$ $\quad -$

$(1-n)\omega$

$\Big\lfloor\ 0$

$\left[\dfrac{\sin(1+n)\pi}{} = -\dfrac{\omega E}{} - \dfrac{\sin(1-n)\pi}{}\right] = 0 \qquad$ for n = 2, 3, 4, ...

$\Big|\Big|\ 2\pi\ \Big\lfloor$

$(1+n)\omega$

$(1-n)\omega\ \Big|$

$$u(t) = \frac{1}{2}a_0 + \sum_{n=2}^{\infty} a_n \cos n\omega t$$

$$u(t) = \frac{E}{\pi} + \frac{E}{2}\sin \omega t - \frac{2E}{\pi}\left(\frac{1}{1.3}\cos 2\omega t + \frac{1}{3.5}\cos 4\omega t + \ldots \right)$$